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Classical Approach to Quantum Theory

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A formulation of nonrelativistic, spinless, quantum mechanics is presented which is based on four postulates. Three of the postulates are very analogous to relations that hold in an operator formulation of classical mechanics, and the fourth is that the wave function evolves linearly in time. The conventional statistical assertions of quantum theory as well as the Schrödinger equation are recovered.

I. INTRODUCTION

At present there are several familiar and well-accepted techniques by means of which one constructs the quantum mechanical description of a classical system. These methods are often associated with the names of von Neumann, Dirac, and Feynman. In order to put the forthcoming discussion into perspective we shall here, very briefly, present the salient features of these approaches: (i) *von Neumann's approach* (von Neumann, 1955): Here, one assumes the customary probability density interpretation for $\psi(\mathbf{x}, t)$ and also that $|(\psi, \varphi_n)|^2$, (where the φ_n are eigenstates of the operator assumed to belong to the observable) gives the probability of finding the system in state φ_n—as well as finding the value of the observable to be the eigenvalue associated with φ_n—upon measurement of the observable. These assumptions are succinctly summarized by the assumption that $\| \hat{E}(I)\psi \|^2$ gives the probability that, in the state ψ, the observable (whose operator has the spectral resolution $\hat{E}$) takes on values in the interval I. Finally, the Schrödinger equation is assumed. (ii) *Dirac's approach* (Dirac, 1947): Here, the emphasis is on operators. One first expresses the classical description of the system canonically. It is again assumed that operators correspond to classical quantities. This correspondence is quantified by the assumption that the Poisson brackets entering the classical

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formulation are to be replaced by (or correspond to) operator commutator brackets. Then the Hamiltonian operator (which is assumed to be the time displacement operator for the wave function) is assumed to be the same function of the operators corresponding to the p's and q's as the classical Hamiltonian is of the momenta and position, thus giving the Schrödinger equation. Finally, it is assumed that the values that any observable quantity can take on are the eigenvalues of the corresponding operator, and the average value of any quantity θ in the state ψ is given by the expression $(\psi, \hat{\theta}\psi)$, where $\hat{\theta}$ is the operator corresponding to θ. (iii) *Feynman's approach* (Feynman and Hibbs, 1965): This approach rests on the notion of a probability amplitude for a particle to go from one space-time event to another. The magnitude squared of this amplitude is assumed to give the probability (density) for the transition. The amplitude itself is assumed to be composed of a sum of contributions, each such having the form $\exp[(i/\hbar)S_{\text{cl}}]$, where S_{cl} is the classical action that characterizes a particle undergoing the transition along a prescribed path. The Schrödinger equation as well as all the customary statistical assertions of quantum theory result.

We comment here that there have also been descriptions or approaches other than those considered above, but the cases cited are sufficient for our purpose.[1]

The purpose of reminding the reader of the above well-known techniques is really to point out that, except for method (iii), the other approaches involve so many disconnected assumptions as certainly not to be considered in any sense derivations of quantum mechanics. More specifically, these approaches give no intuitive reason for associating operators with quantities; provide no such reason or context for their statistical assertions; and finally, do not link the statistical assertions to any derivation of the Schrödinger equation. The only feature operating seems to be the expedient one that "it works." In contrast, method (iii) depends on just a few assumptions which, for the most part, have a connected physical significance relating, to some extent anyway, to ideas in classical physics. This method, in the opinion of the author, amounts to a derivation of quantum mechanics. The fact that the physical meaning of the sum over histories is only qualitatively transparent does not detract from this judgment.

The purpose of the present work is to offer another derivation of nonrelativistic quantum mechanics (of spinless systems), similar in rigor and intuitive appeal (but probably not in scope) to method (iii).

[1]See, for example, Loinger (1962); von Neumann (1932); Uhlhorn (1956); Jordan and Sudarshan (1961).

In the following sections we present the quantum formulation for nonrelativistic (spinless) systems as an analogy to that for classical systems.

As the entire formulation relies very heavily on the author's recent operator formulation of classical mechanics (Cohn, 1980), we begin by reviewing this description of classical mechanics.

2. OPERATOR FORMULATION OF CLASSICAL MECHANICS

To begin with we envisage a set of replicas of the classical system of interest or, equivalently, consider a collection of noninteracting particles moving in the prescribed potential field $V(\mathbf{x})$ present in the system of interest. Ordinary particle classical mechanics is described by the Hamilton–Jacobi equation as well as the mass conservation law which, in our case, takes the form

$$\frac{1}{2m}(\nabla S)^2 + V(\mathbf{x}) = -S_{,t} \tag{1}$$

and

$$\nabla \cdot (\rho^2 \nabla S) = -m\rho^2_{,t} \tag{2a}$$

with

$$\mathbf{p} = \nabla S \quad \text{and} \quad E = -S_{,t} \tag{2b}$$

Here, S is Hamilton's principal function, and ρ^2 is the density (normalized to unity for convenience) in *configuration* space of the noninteracting replicas already mentioned.

These equations can be conveniently summarized by introducing a function, $\psi_{\text{cl}} = \rho \exp[(i/\beta)S]$ [where $(\psi, \psi) = \int \psi^* \psi \, d\mathbf{x} = \text{const} = 1$] and the operator,

$$\hat{\mathcal{H}}_{\text{cl}} \equiv \frac{-\beta^2}{2m} \nabla^2 + V + \beta^2 \hat{\theta}$$

where

$$\hat{\theta}\psi = \frac{1}{2m} \frac{\nabla^2 \rho}{\rho} \psi$$

for any ψ, and where β denotes an unspecified constant of dimension (momentum$\times$length).[2]

Then the single equation

$$\hat{\mathcal{H}}_{\rm cl}\psi_{\rm cl} = i\beta\partial_t\psi_{\rm cl} \tag{3}$$

implies equations (1) and (2a).

Furthermore, we discover the following facts concerning averages over the classical "wave packet" ψ (where $\rho\to 0$ sufficiently fast at infinity), where the brackets $\langle\ \rangle$ surrounding a quantity indicate the average—viz., $\langle\mathbf{p}\rangle = \int\rho^2\mathbf{p}\,d\mathbf{x} = (\psi, \mathbf{p}\psi)$

$$\langle f(\mathbf{x})\rangle = \left(\psi_{\rm cl}, f(\mathbf{x})\psi_{\rm cl}\right) \tag{4a}$$

for any function $f(\mathbf{x})$.

$$\langle\mathbf{p}\rangle = \langle\nabla S\rangle = \left(\psi_{\rm cl}, -i\beta\nabla\psi_{\rm cl}\right) \equiv \left(\psi_{\rm cl}, \hat{\mathbf{p}}\psi_{\rm cl}\right) \tag{4b}$$

and

$$\langle\mathbf{L}\rangle = \langle\mathbf{r}\times\mathbf{p}\rangle = \left(\psi_{\rm cl}, \hat{\mathbf{L}}\psi_{\rm cl}\right) \tag{4c}$$

where we define the momentum operator as $\hat{\mathbf{p}} = -i\beta\nabla$, and the angular momentum operator as $\hat{\mathbf{L}} = -i\beta\mathbf{r}\times\nabla$.

$$\langle\mathbf{p}\rangle = m\frac{d}{dt}\langle\mathbf{x}\rangle = m\frac{d}{dt}\left(\psi_{\rm cl}, \mathbf{x}\psi_{\rm cl}\right) \tag{5}$$

$$\frac{d}{dt}\langle\mathbf{p}\rangle = \langle -\nabla V\rangle \tag{6}$$

$$\langle\mathbf{p}^2\rangle = \left(\psi_{\rm cl}, \hat{\mathbf{p}}^2\psi_{\rm cl}\right) + 2m\beta^2\left(\psi_{\rm cl}, \hat{\theta}\psi_{\rm cl}\right) \tag{7}$$

$$\left(\psi_{\rm cl}, \hat{\mathcal{H}}_{\rm cl}\psi_{\rm cl}\right) = \langle E\rangle = \frac{1}{2m}\langle\mathbf{p}^2\rangle + \langle V\rangle \tag{8}$$

$$(\Delta p_x)^2(\Delta x)^2 \geqslant \beta^2/4 + 2m\beta^2(\Delta x)^2\left(\psi_{\rm cl}, \hat{\theta}\psi_{\rm cl}\right) \qquad (\text{for } x, y, \text{ or } z) \tag{9}$$

where, for example, $(\Delta p_x)^2 \equiv \langle(p_x - \langle p_x\rangle)^2\rangle$, and denotes the dispersion in p_x in the state $\psi_{\rm cl}$.

[2] In the paper by the author where this was first discussed, β was taken equal to 1, and so did not appear as such in the equations. We bring out β explicitly in this discussion because of its relevance to the later quantum mechanical discussion.

We comment that the parameter β is arbitrary, as it remains undetermined by any of the preceding equations.

It turns out, as we shall now discuss, that equations (4b), (4c), and (8) can be expressed in a more unified manner, which will be useful for our purposes. We begin by defining the time transformation (translation) operator, $\hat{T} = \exp(t\partial_t)$, the displacement operator, $\hat{D}_x = \exp(\alpha_x \partial_x)$ (and similarly $\hat{D}_y$ and $\hat{D}_z$), and the rotation operator, $\hat{R}_x = \exp[\theta_x(z\partial_y - y\partial_z)]$ (and similarly $\hat{R}_y$ and $\hat{R}_z$). Beginning with $\hat{T}$ we note that, for small t, $\hat{T} \cong \hat{I} + t\partial_t$, so that we have the relation

$$\left(\psi_{\mathrm{cl}}, \hat{T}\psi_{\mathrm{cl}}\right) = \left(\psi_{\mathrm{cl}}, \left[\hat{I} - \frac{i}{\beta} t\hat{\mathcal{H}}_{\mathrm{cl}}\right]\psi_{\mathrm{cl}}\right) = \langle e^{-(i/\beta)tE}\rangle \qquad \text{(to first order in } t\text{)} \tag{10}$$

where equations (3) and (8) have also been used.

In exactly the same way we obtain the additional relations

$$\left(\psi_{\mathrm{cl}}, \hat{D}_x\psi_{\mathrm{cl}}\right) = \langle e^{(i/\beta)\alpha_x p_x}\rangle \qquad \text{(to first order in } \alpha_x\text{)} \tag{11}$$

and

$$\left(\psi_{\mathrm{cl}}, \hat{R}_x\psi_{\mathrm{cl}}\right) = \langle e^{(i/\beta)\theta_x L_x}\rangle \qquad \text{(to first order in } \theta_x\text{)} \tag{12}$$

Now, let $\mathcal{P}_E(\lambda)$, $\mathcal{P}_{p_x}(\lambda)$, $\mathcal{P}_{L_x}(\lambda)$, respectively, denote the probability densities (at λ) for finding the energy E, the x component of momentum p_x, or the x component of angular momentum L_x. We comment that these probability densities would be determined from ψ_{cl}; for example,

$$\mathcal{P}_E(\lambda) = \lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda} \int_{R(\Delta\lambda)} |\psi|^2(-S_{,t})\, d^3x$$

where $R(\Delta\lambda)$ is the region in configuration space where $\lambda \leqslant -S_{,t} \leqslant \lambda + \Delta\lambda$.

Now then,

$$\langle e^{-(i/\beta)tE}\rangle = \int e^{-(i/\beta)t\lambda}\mathcal{P}_E(\lambda)\, d\lambda = \tilde{\mathcal{P}}_E(t) \tag{13}$$

where $\tilde{\mathcal{P}}_E$ is the Fourier transform of $\mathcal{P}_E$. Similarly, we also have the relations

$$\langle e^{-(i/\beta)\alpha_x p_x}\rangle = \tilde{\mathcal{P}}_{p_x}(\alpha_x) \quad \text{and} \quad \langle e^{-(i/\beta)\theta_x L_x}\rangle = \tilde{\mathcal{P}}_{L_x}(\theta_x) \tag{14}$$

Combining, say, equations (13) and (10), we then have

$$(\psi_{\rm cl}, \hat{T}\psi_{\rm cl}) = \tilde{\mathcal{P}}_{E(t)} \qquad \text{(to first order in } t\text{)} \tag{15}$$

And similarly, from equations (11), (12) and (14) we have

$$(\psi_{\rm cl}, \hat{D}_x\psi_{\rm cl}) = \tilde{\mathcal{P}}_{p_x}(-\alpha_x) \qquad \text{(to first order in } \alpha_x\text{)} \tag{16}$$

and

$$(\psi_{\rm cl}, \hat{R}_x\psi_{\rm cl}) = \tilde{\mathcal{P}}_{L_x}(-\theta_x) \qquad \text{(to first order in } \theta_x\text{)} \tag{17}$$

So, we see that in *classical* mechanics matrix elements of the fundamental operators $\hat{T}, \hat{\mathbf{D}}, \hat{\mathbf{R}}$ are related, for very small values of the involved variables, to the Fourier transform of the probability distribution functions for the observables conjugate to these variables. These relations will be very important in the next section.

3. NONRELATIVISTIC QUANTUM MECHANICS

In this section we develop our formulation of nonrelativistic (spinless) quantum mechanics in analogy to the operator formulation above for classical mechanics. We do this by presenting several postulates, and exploring their consequences.

3.1. Postulates. (P0). Quantum mechanics, like classical mechanics, is described by a complex function, $\psi = r\exp[(i/\beta)\varphi]$ (where β is unspecified), where $(\psi, \psi) = \int \psi^* \psi \, d\mathbf{x} = \text{const} = 1$, and where $\psi^*\psi$ has the significance of being the probability distribution function governing measurements of $\mathbf{x}$ in the state ψ.

(PI). *Relations* in classical mechanics involving only expectation values of observable quantities (like $\mathbf{x}, E, \mathbf{L}, \mathbf{p}$) have the same *form* in quantum mechanics. Expectation values in quantum mechanics are interpreted as averages over repeated measurements on identically prepared systems.[3]

[3]Thus, for example, we assume that the relations equations (4a), (5), (6), and (8) (involving only the last three terms) hold in quantum mechanics. Note that, as yet, we have not indicated how expectation values are to be calculated in quantum mechanics—except for the case of a function of $\mathbf{x}$. Also note that, even though $(d/dt)\langle \mathbf{x} \rangle = \langle d\mathbf{x}/dt \rangle$ is valid in classical mechanics, we do not assume its validity in quantum mechanics as the last term above does not exist then. Thus, we could add a qualifier to (PI) to the effect that classical relations between expectation values are also valid in quantum mechanics *provided the expectation values exist.*

(PII). The relations between the transform of the probability distribution functions and certain matrix elements, which hold only for small values of the associated variables in classical mechanics (viz., equations (15), (16), and (17)) hold for *all* values of the variables in quantum mechanics. That is

$$\tilde{\mathcal{P}}_\theta(\lambda) = (\psi, \hat{\pi}\psi)$$

for *all* λ, where the triplet $\theta, \lambda, \hat{\pi}$ is, respectively, $E, t, \hat{T}$; $p_x, -\alpha_x, \hat{D}_x$ (etc.); and $L_x, -\theta_x, \hat{R}_x$ (etc.).

(PIII). The time evolution of ψ is governed by a *linear* operator.

3.2. Consequences. In this subsection we explore the consequences of the above postulates and recover the conventional formulation of nonrelativistic quantum mechanics.

From (PIII) and (P0) we have

$$(\psi(0), \psi(0)) = (\psi(t), \psi(t)) = (\hat{T}\psi(0), \hat{T}\psi(0)) \tag{18}$$

which implies that $\hat{T}$ is unitary. Then by Stone's theorem (Stone, 1932), we have that

$$\hat{T} = e^{-(i/\beta)\hat{\mathcal{K}}t} \tag{19}$$

where $\hat{\mathcal{K}}$ is Hermitean (assumed independent of t), and the β here is the same as that entering the postulates.

By considering small t, the above equation yields the relation

$$\hat{\mathcal{K}}\psi = i\beta\partial_t\psi \tag{20}$$

Before we find the explicit form of $\hat{\mathcal{K}}$, we first consider other consequences. Now, from (PII) and expressing $\tilde{\mathcal{P}}_E(t)$ in terms of $\mathcal{P}_E(\lambda)$ via the Fourier transform, we have that

$$(\psi, \hat{T}\psi) = \langle e^{-(i/\beta)tE} \rangle \qquad \text{(for all } t\text{)} \tag{21}$$

Expanding both sides of this relation in powers of t and equating coefficients of like powers then gives the relation

$$(\psi, \hat{\mathcal{K}}^n\psi) = \langle E^n \rangle \qquad \text{(for all integer } n \geqslant 0\text{)} \tag{22}$$

In the same way we also obtain the relations

$$(\psi, \hat{p}_x^n \psi) = \langle p_x^n \rangle \quad \text{and} \quad (\psi, \hat{L}_x^n \psi) = \langle L_x^n \rangle \tag{23}$$

for integer $n \geq 0$.

Therefore, for any observable σ ($= \mathbf{x}, \mathbf{p}, \mathbf{L}, E$), with operator $\hat{\sigma}$, we have the following expression for the dispersion in σ for the state ψ:

$$(\Delta\sigma)^2 \equiv \langle (\sigma - \langle \sigma \rangle)^2 \rangle = \left(\psi, (\hat{\sigma} - \langle \sigma \rangle)^2 \psi\right) \tag{24}$$

Thus, $\Delta\sigma = 0$ iff $\hat{\sigma}\psi = \langle \sigma \rangle \psi = \sigma\psi$. Therefore,[4]

$$\text{The values of the observable } \sigma \text{ are the eigenvalues of } \hat{\sigma} \tag{25}$$

Further, if the eigenfunctions of $\hat{\sigma}$ are complete (as they are for $\mathbf{x}, \hat{\mathbf{p}}, \hat{\mathbf{L}}$ and customarily for $\hat{\mathcal{H}}$) then we can express any ψ as

$$\psi = \sum_{\sigma} a_\psi^\sigma \varphi_\sigma \tag{26}$$

where $\hat{\sigma}\varphi_\sigma = \sigma\varphi_\sigma$ [and the φ_σ are orthonormal since $\hat{\sigma}$ ($= \mathbf{x}, \hat{\mathbf{p}}, \hat{\mathbf{L}}, \hat{\kappa}$) is Hermitean]. Of course, if the spectrum of $\hat{\sigma}$ is continuous, the above sum is to be replaced by an integral. Then we have

$$\langle \sigma^n \rangle_\psi = (\psi, \hat{\sigma}^n \psi) = \int |a_\psi^\sigma|^2 \sigma^n \, d\sigma \tag{27}$$

for all integer n, which then implies that

$$|(\psi, \varphi_\sigma)|^2 = |a_\psi^\sigma|^2 = \text{probability (density) of measuring}$$
$$\text{the value } \sigma \text{ in state } \psi \tag{28}$$

We now obtain the explicit form for $\hat{\mathcal{H}}$ easily as follows. From (PI) and equation (8) we have

$$\langle E \rangle = \langle \mathbf{p}^2/2m \rangle + \langle V \rangle \tag{29}$$

[4] This conclusion involves the experimental observation that suitably repeated measurements of a given observable yield identical results.

and, therefore, by equations (22) and (23) we have that

$$(\psi, \hat{\mathcal{H}}\psi) = \left(\psi, \frac{1}{2m}\hat{\mathbf{p}}^2\psi\right) + (\psi, V\psi) \tag{30}$$

for all ψ. Therefore, we have that

$$\hat{\mathcal{H}} = \frac{1}{2m}\hat{\mathbf{p}}^2 + V(\mathbf{x}) \tag{31}$$

which, together with equation (20) is Schrödinger's equation.

Finally, we discuss the parameter β. As already mentioned in Section 2 the parameter β is not fixed by the requirements of classical mechanics. Put another way, the Hamilton–Jacobi equation and the equation of mass conservation result from equation (3) regardless of the value of β. However, the quantum mechanical situation is different. Here, we have that the values of an observable σ are the eigenvalues of $\hat{\sigma}$. And since the $\hat{\sigma}$ $(= \hat{\mathbf{p}}, \hat{\mathbf{L}}, \hat{\mathcal{H}})$ contain β, the eigenvalues will depend on β, and thus β may be found by comparison with experiment. A more dramatic example of this is given by the relation, say,

$$(\Delta p_x)^2(\Delta x)^2 \geqslant \beta^2/4 \tag{32}$$

which one derives from the quantum formalism in the usual manner. [Note that the classical relation, equation (9) is not assumed to hold in quantum mechanics as its involves quantities other than expectation values of observable quantities.] Unlike the classical case, where $(\Delta p_x)^2(\Delta x)^2$ had no positive lower bound [because of the *negative* term, $(\psi_{\text{cl}}, \hat{\theta}\psi_{\text{cl}})$, appearing in the expression] in the quantum case there is a lower bound determined by β and amenable to experimental observation. At this point then we are justified in identifying β with $\hbar$.

We see then that we have recovered the conventional formulation of quantum mechanics from a unified and fairly intuitive viewpoint. Of course, in order to apply the formulation to a very broad range of problems we need the additional assumption that, in general, observable operators have complete sets of eigenfunctions, as well as a rule of correspondence allowing one to construct the operators corresponding to more elaborate functions of $\mathbf{x}, \mathbf{p}, \mathbf{L}, \ldots$ etc. These grace notes, however, are considered to be enrichments on the main theme.

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